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1876
Johns Hopkins University
of the

On Certain Chains of Theorems in Reflexive Geometry

Dissertation

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by

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Baltimore

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On Certain Chains of Theorems in Reflexive Geometry

§ 1. Introduction

It is of interest to extend properties of the triangle to more than three lines, and when possible, to n lines of a plane. This can often be done by a process, called by Professor Frank Morley, geometrical interpolation. As an illustration, I consider, in what follows, a problem proposed by Deslonchamps in his "Questions de Géométrie Élémentaire"* namely: "Les trois sommets a, b, c d'un triangle on abaisse des perpendiculaires ap, bq, cr sur une droite quelconque de son plan, puis, des points p, q, r des perpendiculaires sur bc, ca, ab: ces trois dernières droites se coupent en un même point." A proof of this problem, by means of tri-coordinates, is given by Kazimierz Zwierzyński in "Archiv der Mathematik und Physik."†

* 1875, page 241, no. 77 † 1901, vol 1, pp 175-180

In the present paper, some extensions of these theorems are considered as indicated in the following scheme:

Undirected Lines

First Chain		Second Chain	
1-a	3 lines	2-a	4 lines
1-b	4 lines	2-b	5 lines
1-c	5 lines	2-c	6 lines
1-g	general statement	2-g	general statement

Directed Lines

I-a	4 lines	II-a	5 lines
I-b	5 lines	II-b	6 lines
I-g	general statement	II-g	general statement

We name a point of the plane by a complex number.

§ 2 - Some Fundamental Formulae in Reflexive Geometry

1- Reflexion in a Line

If the two triangles x, z, b and y, b, z are con-

versely similar, and y is the reflexion of x, b the reflexion of x, and a the reflexion of b, in one and the same line, then the two triangles x, a, b and $\bar{y}, \bar{b}, \bar{a}$ are directly similar.

The condition that any two triangles x_1, x_2, x_3 and y_1, y_2, y_3 be directly similar is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0 \qquad (1)$$

therefore, since the two triangles x, a, b and $\bar{y}, \bar{b}, \bar{a}$ are directly similar,

$$\begin{vmatrix} x & a & b \\ \bar{y} & \bar{b} & \bar{a} \\ 1 & 1 & 1 \end{vmatrix} = 0 \qquad (2)$$

Expanding the determinant, we obtain

$$\frac{x}{(a-b)} + \frac{\bar{y}}{(\bar{a}-\bar{b})} = \frac{a\bar{a} - b\bar{b}}{(a-b)(\bar{a}-\bar{b})} \qquad (3)$$

which can be written more symmetrically, as

$$\frac{x-b}{a-b} + \frac{\bar{y}-\bar{b}}{\bar{x}-\bar{b}} = 1 \qquad (4)$$

or

$$\frac{x-a}{a-b} + \frac{\bar{y}-\bar{a}}{\bar{x}-\bar{b}} = -1 \qquad (5)$$

If $b = 0 = \bar{b}$; (4) becomes

$$\frac{x}{a} + \frac{\bar{y}}{\bar{a}} = 1 \qquad (6)$$

which is the standard equation for reflexion in a line.

When y is x, this equation (6) becomes the equation of the straight line, in which $\underline{a}$ is the image of the base-point. But we prefer to establish the equation also as follows:

2- Equation of a Straight Line

The base circle is

$$x = t \qquad (1)$$

Let (1) carries with it, its conjugate equation

$$\bar{x} = 1/t \qquad (2)$$

combining (1) and (2) we obtain

$$x\bar{x} = 1 \qquad (3).$$

which is the self-conjugate equation of the unit circle.

In rectangular coordinates, the unit circle is expressed by the equation

$$x^2 + y^2 = 1 \qquad (4)$$

If now, we employ the circular coordinates, $x = X + iY$; $\bar{x} = X - iY$; instead of the rectangular coordinates, it is readily seen that (4) becomes (3)

Polarizing (3) with respect to the points $y, \bar{y}$ and adding, we get the self-conjugate equation

$$x\bar{y} + \bar{x}y = 2 \qquad (5)$$

Suppose $y = t/\rho$; $\bar{y} = 1/\rho t$; then we have, substituting these values in (5)

$$x/t + \bar{x}t = 2\rho \qquad (6)$$

as the equation of any line, and $\rho = 0$ gives us,

$$x + t^2\bar{x} = 0 \qquad (7)$$

as the equation of any straight line through the origin

But (6) can be written as

$$x + t^2\bar{x} = 2\rho t \qquad (8)$$

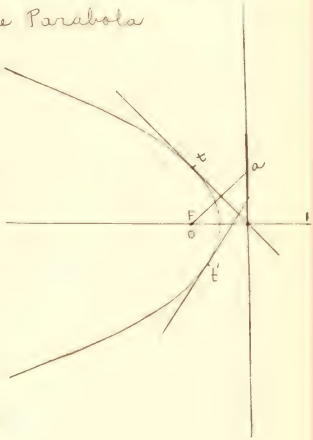
or
$$\frac{x}{2pt} + \frac{t\,\bar{x}}{2p} = 1 \qquad (9)$$

If now, we let $2p\,t = a$; then $2p/t = \bar{a}$, and (9) becomes,

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \qquad (10)$$

which is the standard equation of a straight line

3- Map- equation of the Parabola



The parabola possesses the property that the reflexion of the focus in a tangent lies on the directrix.

Let O be the focus of the parabola; a the reflexion of F in a tangent, and let 1 be the reflexion of F in the directrix; then it is easily seen that

$$t\,x = a - x \qquad (1)$$

$$x = \frac{a}{1+t} \qquad (2)$$

and also that

$$t'a = 1 - a \qquad (3)$$

$$a = \frac{1}{1+t'} \qquad (4)$$

Substituting this value of a in (2) we obtain

$$x = \frac{1}{(1+t)(1+t')} \qquad (5)$$

This represents for a given t or t' the tangent of the parabola.

We speak of the line t, or the point t of a parabola, as is convenient.

By putting $t = t'$ we obtain the envelope, namely,

$$x = \frac{1}{(1+t)^2} \qquad (6)$$

the map-equation of the parabola.

4. Map-equation of the Double-Parabola*

The map-equation of the double parabola is

* The name is due to Clifford.

$$x = \frac{A_1}{(\alpha_1 - t)^2} + \frac{A_2}{(\alpha_2 - t)^2} \qquad (1)$$

(where α_1, α_2 and t are turns or orthogonal numbers) provided the cusp condition $\frac{dx}{dt} = 0$, is satisfied for t a turn.

Applying the cusp condition to (1), we have,

$$\frac{dx}{dt} = \sum^{2} \frac{2A_i}{(\alpha_i - t)^3} = 0 \qquad (2)$$

The conjugate equation of (2) is

$$\sum^{2} \frac{2\bar{A}_i \alpha_i^3 t^3}{(\alpha_i - t)^3} = 0 \qquad (\bar{2})$$

Hence, the condition that (1) has a cusp is satisfied when

$$A_i = \bar{A}_i \alpha_i^3 \qquad (3)$$

And in general, the map-equation of an n-fold parabola is

$$x = \sum^{n} \frac{A_i}{(\alpha_i - t)^2} \qquad (4)$$

provided the cusp condition $dx/dt = 0$ is satisfied for a turn t, that is when (3) is satisfied for $i = 1, 2 \ldots n$.

5- The Equation of a Directed Line

Directed lines are lines which possess a right and left handside, and therefore should be marked with an arrow head.

The normal form of a line, in rectangular coordinates, is

$$x \cos\alpha + y \sin\alpha = p \qquad (1)$$

where p is the $\perp$ distance from the line to the origin.

If now, we employed the circular coordinates, $x = x + iy$; $\bar{x} = x - iy$; instead of the rectangular coordinates, (1) becomes

$$(x + \bar{x}) \cos\alpha + \left(\frac{x - \bar{x}}{i}\right) \sin\alpha = 2p \qquad (2)$$

or

$$x(\cos\alpha - i \sin\alpha) + \bar{x}(\cos\alpha + i \sin\alpha) = 2p \qquad (3)$$

But $(\cos\alpha - i \sin\alpha)$ is a term, let us represent it by t; then $(\cos\alpha + i \sin\alpha) = 1/t$

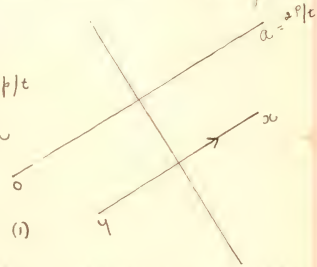
Putting these values in (3) we have

$$xt + \bar{x}/t = 2p \qquad (4)$$

as the equation of a directed line.

6 - Distance from a Straight Line

In the case of directed lines, distance = length ÷ direction.



Let o be the reflexion of $a = 2p/t$ (where p is real) in the axis; then the equation of the axis is

$$y t + \bar{x}/t = 2p \qquad (1)$$

From the figure, we see that the distance of the point x from the axis is

$$\frac{x - y}{2 \cdot 1/t} \qquad (2)$$

But from (1) we have that

$$t y = -\bar{x}/t + 2p \qquad (3)$$

Therefore,

$$\frac{x - y}{2 \cdot 1/t} = \frac{1}{2}\left(t x + \bar{x}/t - 2p\right) \qquad (4)$$

Thus, twice the distance of a point x, from a line is expressed by the equation

$$2\,\underline{|}\,| = tx + {}^{\bar{x}}/t - 2p \qquad (5)$$

Undirected Lines

§3.7 First Chain of Theorems

1-a Given three lines λ_1, λ_2, and λ_3 tangent to the parabola

$$x = \frac{1}{(1+t)^2} \qquad (1)$$

at the points t_1, t_2 and t_3 respectively, then the equation of λ_i (for $i = 1, 2, 3$) will be

$$x = \frac{1}{(1+t_i)(1+t)} \qquad (2)$$

The intersection of λ_1 and λ_2 is given by the equation

$$x_{12} = \frac{1}{(1+t_1)(1+t_2)} \qquad (3)$$

for if we let $t = t_2$ in the equation of λ_1, it is identical with (5) ∴ (3) is the equation of a point on the line λ_1, and similarly, if we let $t = t_1$ in the equation of λ_2, the equation becomes the same as (3) ∴ (3) is the equation of a point on the line λ_2; consequently (3) is the

equation of the intersection of the two lines λ_1 and λ_2. In like manner, we obtain the intersections of λ_2 and λ_3; and λ_3 and λ_1; namely

$$x_{23} = \frac{1}{(1+t_2)(1+t_3)} \qquad (4)$$

$$x_{31} = \frac{1}{(1+t_3)(1+t_1)} \qquad (5)$$

These equations carry with them, their conjugates;

$$\bar{x}_{12} = \frac{t_1 t_2}{(1+t_1)(1+t_2)} \qquad (6)$$

$$\bar{x}_{23} = \frac{t_2 t_3}{(1+t_2)(1+t_3)} \qquad (7)$$

$$\bar{x}_{31} = \frac{t_3 t_1}{(1+t_3)(1+t_1)} \qquad (8)$$

We shall now proceed to reflect these intersections x_{12}, x_{23} and x_{31} in an arbitrary line, say, the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \qquad (9)$$

From section 1, we see that reflexion in the line is

given by the equation

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \qquad (10)$$

Therefore, reflecting the point x_{12} (the intersection of λ_1 and λ_2) whose coordinates are

$$x_{12} = \frac{1}{(1+t_1)(1+t_2)} \quad ; \quad \bar{x}_{12} = \frac{t_1 t_2}{(1+t_1)(1+t_2)} \qquad (11)$$

we obtain, as its reflexion in the line, the point

$$\frac{x}{a} = 1 - \frac{t_1 t_2}{\bar{a}\,(1+t_1)(1+t_2)} \qquad (12)$$

its conjugate, $$\frac{\bar{x}}{\bar{a}} = 1 - \frac{1}{a(1+t_1)(1+t_2)} \qquad (13)$$

Similarly, we obtain the reflexions of the points x_{23} and x_{31} in the line, namely

$$\frac{x}{a} = 1 - \frac{t_2 t_3}{\bar{a}\,(1+t_2)(1+t_3)} \qquad (14)$$

$$\frac{\bar{x}}{\bar{a}} = 1 - \frac{1}{a(1+t_2)(1+t_3)} \qquad (15)$$

and $$\frac{x}{a} = 1 - \frac{t_3 t_1}{\bar{a}\,(1+t_3)(1+t_1)} \qquad (16)$$

$$\frac{\bar{x}}{\bar{a}} = 1 - \frac{1}{a(1+t_1)(1+t_1)} \qquad (17)$$

The equation of a tangent to the parabola at the point t_1, is

$$x = \frac{1}{(1+t_1)(1+t)} \qquad (18)$$

and its conjugate equation is

$$\bar{x} = \frac{t_1 t}{(1+t_1)(1+t)} \qquad (19)$$

Adding (18) and (19) we have

$$x + \bar{x} = \frac{1}{(1+t_1)(1+t)} + \frac{t_1 t}{(1+t_1)(1+t)} \qquad (20)$$

or

$$x + \frac{\bar{x}}{t_1} = \frac{1}{1+t_1} \qquad (21)$$

which is a self-conjugate equation of the tangent to the parabola at the point t_1. A line perpendicular to this tangent will be of the form

$$x - \frac{\bar{x}}{t_1} = c - \frac{\bar{c}}{t_1} \qquad (22)$$

Consequently, the equation of a line on the reflexion of x_{23} and perpendicular to the line λ_1, which is tangent to the parabola at the point t_1 is

$$x - \frac{\bar{x}}{t_1} = a - \frac{a t_2 t_3}{\bar{a}(1+t_2)(1+t_3)} - \frac{1}{t_1}\left[\bar{x} - \frac{\bar{x}}{a(1+t_2)(1+t_3)}\right] \qquad (23)$$

For convenience, let us write (23) in the following manner

$$t_1 x - \bar{x} = a t_1 - \frac{a t_1 t_2 t_3 (1+t_1)}{\bar{a}(1+t_1)(1+t_2)(1+t_3)} - \bar{x} + \frac{\bar{x}(1+t_1)}{a(1+t_1)(1+t_2)(1+t_3)} \qquad (24)$$

or

$$t_1 x - \bar{x} = a t_1 - \frac{a S_3 (1+t_1)}{\bar{a}\,\overset{3}{\Pi}} - \bar{a} + \frac{\bar{a}(1+t_1)}{a\,\overset{3}{\Pi}} \qquad (25)$$

where $\overset{3}{\Pi} = (1+t_1)(1+t_2)(1+t_3)$; $S_3 = t_1 t_2 t_3$.

Let us replace t_1 in (25) by the parameter t, and we obtain the equation

$$t x - \bar{x} = a t - \frac{a S_3 (1+t)}{\bar{a}\,\overset{3}{\Pi}} - \bar{a} + \frac{\bar{a}(1+t)}{a\,\overset{3}{\Pi}} \qquad (26)$$

Now then, if we let $t = t_2$, it is easily seen that (26) becomes the equation of a line on the reflex-

ion of x_{13} and perpendicular to the line λ_2, similarly when $t = t_3$; (26) represents the line on the reflexion of x_{12} and perpendicular to the line λ_3.

Thus, by means of a process called "interpolation", we are enabled to write one equation, which as the parameter "t" assumes the values t_1, t_2, t_3 gives us the three lines u_1, u_2, and u_3. Therefore, the three lines u_1, u_2 and u_3 must intersect in a point, and their point of intersection is expressed by the equation

$$x_{123} = x - \frac{u S_3}{\bar{u}\,\overset{3}{\Pi}} + \frac{\bar{u}}{u\,\overset{3}{\Pi}} \qquad (27)$$

Hence we have the theorem of* above: Given three lines and an extra line; if we reflect the vertices of the three lines, in the arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, the three perpendiculars will meet in a point.

We will call this point the associated point of

* page 241, no. 77

the line λ, or p_3, with reference to the 11 regular lines λ_1, λ_2, λ_3.

1-b Again, let us consider four undirected lines λ_1, λ_2, λ_3 and λ_4 as tangents to the parabola

$$x = \frac{1}{(1+t)^2} \qquad (1)$$

We will recall that λ_i (for $i = 1, 2, 3, 4$) is expressed by the equation

$$x = \frac{1}{(1+t_i)(1+t)} \qquad (2)$$

For many purposes the circumcenter of a 3-line plays the part of the intersection of a 2-line. We will now find the circumcenter of three tangents to the parabola. First let us examine the intersections of the 3-line formed by $\lambda_1, \lambda_2, \lambda_3$. On page 11, we defined the intersection of two tangents λ_i and λ_j as follows

$$x_{ij} = \frac{1}{(1+t_i)(1+t_j)} \qquad (3)$$

Therefore, the intersections of λ_1, λ_2; $\lambda_2 \lambda_3$ and λ_3, λ_1 have as their respective equations

$$x_{12} = \frac{1}{(1+t_1)(1+t_2)} \qquad (4)$$

$$x_{23} = \frac{1}{(1+t_2)(1+t_3)} \qquad (5)$$

$$x_{31} = \frac{1}{(1+t_3)(1+t_1)} \qquad (6)$$

By means of interpolation, we are enabled to write an equation that will pick up all three of these points; such an equation is

$$x = \frac{(1+t)}{(1+t_1)(1+t_2)(1+t_3)} \qquad (7)$$

for when $t = t_1$; (7) reduces to (5), and therefore (7) passes thru the intersection of λ_2 and λ_3; that is (7) is on the point x_{23}; when $t = t_2$; (7) reduces to (6), and is therefore on x_{13}; and finally when $t = t_3$; (7) reduces to (4) and passes thru the point x_{12}. Consequently, (7) represents a curve which passes

thru the three points x_{12}, x_{23} and x_{31}.

What then, is the nature of the locus defined by (7)? The map-equation of a circle in plane reflexive geometry is of the form

$$x = a + b\,t \qquad (8)$$

where a = the center; and $|b|$ = the radius of the circle. In this wise, the map-equation of a circle is linear in t. It is obvious, that (7) is a form analogous to (8), therefore (7) is the map-equation of a circle, whose center is

$$x_{123} = \frac{1}{(1+t_1)(1+t_2)(1+t_3)} \qquad (9)$$

and the conjugate equation is

$$\overline{x}_{123} = \frac{S_3}{(1+t_1)(1+t_2)(1+t_3)} \qquad (10)$$

This is then the circumcenter. Therefore, associated with four undirected lines, we will have four circumcenters, one for each 3-line.

We shall now proceed, as we did in section 1-a

page 12, first we will reflect the circumscribed x_{123} in an arbitrary line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \qquad (11)$$

and then we will erect a perpendicular to λ_4, on this reflexion. Since reflexion in a line is given by

$$\frac{x}{a} + \frac{\bar{y}}{\bar{b}} = 1 \qquad (12)$$

we will obtain as the reflexion of x_{123} in the line (11)

$$\frac{x}{a} = 1 - \frac{1}{a}\left[\frac{S_3}{(1+t_1)(1+t_2)(1+t_3)}\right] \qquad (13)$$

and its conjugate equation

$$\frac{\bar{x}}{\bar{a}} = 1 - \frac{1}{a}\left[\frac{1}{(1+t_1)(1+t_2)(1+t_3)}\right] \qquad (14)$$

Now since λ_4 is a tangent to the parabola

$$x = \frac{1}{(1+t)^2} \qquad (15)$$

at the point t_4, we can write its equation as

$$x + \frac{\bar{x}}{t_4} = \frac{1}{1+t_4} \qquad (16)$$

A line $\perp$ to (16) will be of the form

$$x - \frac{\bar{x}}{t_4} = c - \frac{\bar{c}}{t_4} \qquad (17)$$

Therefore, the equation of a line perpendicular to λ_4, and on the reflexion of the point x_{123}, in the arbitrary line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \qquad (18)$$

will have the following equation, namely

$$x - \frac{\bar{x}}{t_4} = a - \frac{a S_3'}{\bar{a}\,\overset{3}{\Pi}} - \frac{\bar{a}}{t_4} + \frac{\bar{a}}{a\,\overset{3}{\Pi}\,t_4} \qquad (19)$$

or $$t_4 x - \bar{x} = a t_4 - \frac{a S_4}{\bar{a}\,\overset{3}{\Pi}} - \bar{x} + \frac{\bar{x}}{a\,\overset{3}{\Pi}} \qquad (20)$$

where $\overset{3}{\Pi} = (1+t_1)(1+t_2)(1+t_3)$; and $S_4 = t_1 t_2 t_3 t_4$; S' [illegible]

Similarly, the equation of a line perpendicular to λ_3, and on the reflexion of the point x_{124}, in the arbitrary line, is

$$x - \frac{\bar{x}}{t_3} = a - \frac{a S_3'}{\bar{a}\,\overset{3}{\Pi}} - \frac{\bar{a}}{t_3} + \frac{\bar{a}}{a\,\overset{3}{\Pi}\,t_3} \qquad (21)$$

$$\text{or} \quad t_3 x - \bar{x} = a t_3 - \frac{a S_4}{\bar{a}\,\overset{3}{\Pi}} - \bar{a} + \frac{\bar{a}}{a\,\overset{3}{\Pi}} \qquad (22)$$

where $\overset{3}{\Pi} = (1+t_1)(1+t_2)(1+t_3)$; $S_4 = t_1 t_2 t_3 t_4$; $S' = t_1 t_2 t_4$

Again, by the process called "interpolation", we are enabled to write an equation that will pick up these four lines, namely

$$t x - \bar{x} = a t - \frac{a S_4 (1+t)}{\bar{a}\,\overset{4}{\Pi}} - \bar{a} + \frac{\bar{a}(1+t)}{a\,\overset{4}{\Pi}} \qquad (23)$$

(where $\overset{4}{\Pi} = (1+t_1)(1+t_2)(1+t_3)(1+t_4)$; $S_4 = t_1 t_2 t_3 t_4$); for when $t = t_4$; (23) reduces to (20) and similarly when $t = t_3$; (23) reduces to (22) etc. Thus, it is easily seen, that (23) picks up the four lines which are perpendicular to λ_1, λ_2, λ_3 and λ_4 respectively, and which also lie respectively, on the reflections of x_{234}, x_{134}, x_{124} and x_{123} in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \qquad (24)$$

Moreover, (23) gives the intersection of these four perpendicular lines. Consequently, the point of

intersection of the four lines is

$$x_{1234} = a - \frac{a S_4}{\bar{a} \overset{4}{\Pi}} + \frac{\bar{a}}{a \overset{4}{\Pi}} \qquad (25)$$

and its conjugate equation is

$$\bar{x}_{1234} = \bar{a} - \frac{\bar{a}}{a \overset{4}{\Pi}} + \frac{a S_4}{\bar{a} \overset{4}{\Pi}} \qquad (26)$$

where $\overset{4}{\Pi} = (1+t_1)(1+t_2)(1+t_3)(1+t_4)$; $S_4 = t_1 t_2 t_3 t_4$

Hence we can state the following theorem: Given four lines and an extra line; if we reflect the circumcentre of the four 3-lines in the arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, the four perpendiculars will meet in a point.

We will call this point p_4.

1-C Now, we shall consider, five undirected lines in the tangents of a double parabola. The equation of the double parabola is

$$x = \frac{A}{(\alpha - t)^2} + \frac{B}{(\beta - t)^2} \qquad (1)$$

provided the cusp condition $\frac{dt}{dt} = 0$; for t a turn. In this case the cusp condition is $\bar{A}\alpha^3 = A$; $\bar{B}\beta^3 = B$.

Let the five lines under consideration be designated as λ_1, λ_2, λ_3, λ_4 and λ_5. The equation of a tangent to the double parabola, at the point t_1, is

$$x = \frac{A}{(\alpha - t_1)(\alpha - t)} + \frac{B}{(\beta - t_1)(\beta - t)} \qquad (2)$$

or in general, the equation of λ_i, $(i = 1, 2, 3, 4, 5)$ is

$$x = \frac{A}{(\alpha - t_i)(\alpha - t)} + \frac{B}{(\beta - t_i)(\beta - t)} \qquad (3)$$

We shall now proceed to find the intersection of two tangents, say λ_1 and λ_2. Since, the equation of λ_1, is

$$x = \frac{A}{(\alpha - t_1)(\alpha - t)} + \frac{B}{(\beta - t_1)(\beta - t)} \qquad (4)$$

and that of λ_2, is

$$x = \frac{A}{(\alpha - t_2)(\alpha - t)} + \frac{B}{(\beta - t_2)(\beta - t)} \qquad (5)$$

then, the equation of x_{12} the intersection of $\lambda_1 \lambda_2$

will be

$$x_{12} = \frac{A}{(\alpha - t_1)(\alpha - t_2)} + \frac{B}{(\beta - t_1)(\beta - t_2)} \qquad (6)$$

Similarly, we can find the other intersections x_{13}, x_{23}, x_{14}, x_{15} etc.

In section 1-b, we found that to each 3-line there was associated a circumcenter; ∴ to every 3-line arising from these five lines, there is associated a circumcenter. Since, we know the equation of the intersection of λ_1, λ_2, we can, by symmetry, write the equations of the intersections of λ_1, λ_3; λ_2, λ_3.

Thus

$$x_{12} = \frac{A}{(\alpha - t_1)(\alpha - t_2)} + \frac{B}{(\beta - t_1)(\beta - t_2)} \qquad (7)$$

$$x_{23} = \frac{A}{(\alpha - t_2)(\alpha - t_3)} + \frac{B}{(\beta - t_2)(\beta - t_3)} \qquad (8)$$

$$x_{31} = \frac{A}{(\alpha - t_3)(\alpha - t_1)} + \frac{B}{(\beta - t_3)(\beta - t_1)} \qquad (9)$$

By means of interpolation, we can get the equation of the circumcircle, which passes through

the points x_{12}, x_{23}, x_{31}; for if $t = t_i$ (where $i = 1, 2, 3$) in the equation

$$x = \frac{A(\alpha - t)}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)} + \frac{B(\beta - t)}{(\beta - t_1)(\beta - t_2)(\beta - t_3)} \tag{10}$$

(10) becomes successively (8), (9) and (7).

Now since (10) is of the form

$$x = a + bt \tag{11}$$

it is a circle. The center of the circle (10) is

$$x_{123} = \frac{A\alpha}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)} + \frac{B\beta}{(\beta - t_1)(\beta - t_2)(\beta - t_3)} \tag{12}$$

But from four lines, we can form four 3-lines; and since, for every 3-line there is a circumcenter; therefore, from the four lines λ_1, λ_2, λ_3 and λ_4, we can obtain the four circumcenters x_{123}, x_{124}, x_{134}, x_{234}. Again, we can write their equations, by means of symmetry; thus:

$$x_{123} = \frac{A\alpha}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)} + \frac{B\beta}{(\beta - t_1)(\beta - t_2)(\beta - t_3)} \tag{13}$$

$$x_{124} = \frac{A\alpha}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_4)} + \frac{B\beta}{(\beta - t_1)(\beta - t_2)(\beta - t_4)} \tag{14}$$

$$x_{134} = \frac{A\alpha}{(\alpha - t_1)(\alpha - t_3)(\alpha - t_4)} + \frac{B\beta}{(\beta - t_1)(\beta - t_3)(\beta - t_4)} \tag{15}$$

$$x_{234} = \frac{A\alpha}{(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)} + \frac{B\beta}{(\beta - t_2)(\beta - t_3)(\beta - t_4)} \tag{16}$$

Symmetrizing, we obtain

$$x = \frac{A\alpha(\alpha - t)}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)} + \frac{B\beta(\beta - t)}{(\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)} \tag{17}$$

which equation, obviously, as t successively assumes the values t_1, t_2, t_3, t_4, becomes respectively, (16), (15), (14), (13). (17), therefore, passes through the points $x_{123}, x_{124}, x_{134}, x_{234}$

But (17) is of the form

$$x = a + bt \tag{18}$$

$\therefore$ (17) is a circle and its center is

$$x_{1234} = \frac{A\alpha^2}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)} + \frac{B\beta^2}{(\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)}. \tag{19}$$

Its conjugate is

$$\bar{x}_{1234} = \frac{\bar{A}\,{}^1/\alpha^2}{({}^1/\alpha - {}^1/t_1)({}^1/\alpha - {}^1/t_2)({}^1/\alpha - {}^1/t_3)({}^1/\alpha - {}^1/t_4)} + \frac{\bar{B}\,{}^1/\beta^2}{({}^1/\beta - {}^1/t_1)({}^1/\beta - {}^1/t_2)({}^1/\beta - {}^1/t_3)({}^1/\beta - {}^1/t_4)} \tag{20}$$

Applying the curve condition $\bar{A}\alpha^3 = A$, $\bar{B}\beta^3 = B$, and letting $P_4 = (\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)$ and $Q_4 = (\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)$, we have

$$\bar{x} = \frac{\bar{A}\alpha^2 S_4}{P_4} + \frac{\bar{B}\beta^2 S_4}{Q_4} \tag{21}$$

$$= \frac{A S_4}{\alpha P_4} + \frac{B S_4}{\beta Q_4} \tag{22}$$

$$= \sum^{2} \frac{A S_4}{\alpha P_4} \tag{23}$$

where $S_4 = t_1 t_2 t_3 t_4$

If now we reflect this point to x_{1234} in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \tag{24}$$

we obtain, as its reflexion, the point

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{A S_4}{\alpha P_4} \tag{25}$$

its conjugate $$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum \frac{A \alpha^2}{P_4} \tag{26}$$

The equation of a tangent to the conic pas-

rabola at the point t_5 is

$$x = \frac{A}{(\alpha - t_5)(\alpha - t)} + \frac{B}{(\beta - t_5)(\beta - t)} \tag{27}$$

and its conjugate equation is

$$\bar{x} = \frac{\bar{A}\,\alpha^2\, t\, t_5}{(\alpha - t_5)(\alpha - t)} + \frac{\bar{B}\,\beta^2\, t\, t_5}{(\beta - t_5)(\beta - t)} \tag{28}$$

Since the conjugation is $\bar{A}\,\alpha^3 = A$; $\bar{B}\,\beta^3 = B$; we can write (28) as

$$\frac{\bar{x}}{t_5} = \frac{A\,t}{\alpha(\alpha - t)(\alpha - t_5)} + \frac{B\,t}{\beta(\beta - t)(\beta - t_5)} \tag{29}$$

Subtracting (29) from (27) we obtain a self-conjugate expression for the tangent to the double parabola at the point t_5, namely,

$$x - \frac{\bar{x}}{t_5} = \frac{A}{\alpha(\alpha - t_5)} + \frac{B}{\beta(\beta - t_5)} \tag{30}$$

or

$$x - \frac{\bar{x}}{t_5} = \sum_1^2 \frac{A}{\alpha(\alpha - t_5)} \tag{31}$$

A line perpendicular to the tangent at the point $\bar{x}_{1234}$ is expressed by the equation

$$x + \frac{\bar{x}}{t_5} = a - \frac{a}{\bar{a}} \sum^{2} \frac{A S_4}{\alpha P_4} + \frac{\bar{a}}{t_5} - \frac{\bar{a}}{a t_5} \sum^{2} \frac{A \alpha^2}{P_4} \qquad (27)$$

$$\text{or,} \quad t_5 x + \bar{x} = a t_5 - \frac{a}{\bar{a}} \sum^{2} \frac{A S_5}{\alpha P_4} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A \alpha^2}{P_4} \qquad (28)$$

where $P_4 = (\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)$; $\Omega_4 = (\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)$ and $S_5 = t_1 t_2 t_3 t_4 t_5$.

But in this case, we will have five perpendicular lines, namely l_1, perpendicular to λ_1 and on $\bar{x}_{2345}$; $l_2 \perp \lambda_2$ and on $\bar{x}_{1345}$; $l_3 \perp \lambda_3$ and on $\bar{x}_{1245}$; $l_4 \perp \lambda_4$ and on $\bar{x}_{1235}$, and $l_5 \perp \lambda_5$ and on the point $\bar{x}_{1234}$.

Symmetrizing so as to pick up these five lines we obtain the equation

$$t x + \bar{x} = a t - \frac{a}{\bar{a}} \sum^{2} \frac{A S_5 (\alpha - t)}{\alpha P_5} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A \alpha^2 (\alpha - t)}{P_5} \qquad (29)$$

It is at once, evident, when $t = t_i$ (where $i = 1, 2, 3, 4, 5$); (29) gives us the line l_i (where $i = 1, 2, 3, 4, 5$)

But (29) is the equation of the intersection of the five lines l_1, l_2, l_3, l_4 and l_5. This point has the equation

$$x_{12345} = a + \frac{a}{\bar{a}} \sum^{2} \frac{A S_5}{P_5} + \frac{\bar{a}}{a} \sum^{2} \frac{A \alpha^2}{P_5} \qquad (30)$$

Consequently, we have the theorem.

Given five lines and an extra line; if we reflect the centric points of the five 4-lines in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, the five perpendiculars will meet in a point.

We will call this point p_5.

In general, we might say:

Given n lines and an extra line; if we reflect the centric points of the n "$(n-1)$-lines" in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, the n perpendiculars will meet in a point

We will call this point p_n.

§4. Second Chain of Theorems

2-a Given four undirected lines λ_1, λ_2, λ_3 and λ_4 as tangents to the parabola

$$x = \frac{1}{(1+t)^2} \qquad (1)$$

at the points t_1, t_2, t_3 and t_4 respectively, then the equation of the line λ_i $(i = 1, 2, 3, 4)$ is

$$x = \frac{1}{(1+t_i)(1+t)} \qquad (2)$$

With four unlinked lines, we can form four 3-lines, for $C_3^4 = 4$. Here, the four 3-lines are composed of $\lambda_1, \lambda_2, \lambda_3$; $\lambda_1, \lambda_2, \lambda_4$; $\lambda_1, \lambda_3, \lambda_4$; and $\lambda_2, \lambda_3, \lambda_4$ respectively. Now, by the theorem of section 1-v, we found that to every 3-line $\lambda_i, \lambda_j, \lambda_k$ there is associated a point x_{ijk} called p_3. Consequently, to the 3-line $\lambda_1, \lambda_2, \lambda_3$ is associated a p_3 point x_{123}; to the 3-line $\lambda_1, \lambda_2, \lambda_4$ the p_3 point x_{124}, and so on.

The equation of the point x_{123} is

$$x_{123} = a - \frac{a\, S_3}{\bar{a}\, \overset{3}{\pi}} + \frac{\bar{a}}{a\, \overset{3}{\pi}} \qquad (3)$$

where $\overset{3}{\pi} = (1+t_1)(1+t_2)(1+t_3)$; $S_3 = t_1 t_2 t_3$

and its conjugate equation is

$$\bar{x}_{123} = \bar{a} - \frac{\bar{a}}{a\, \overset{3}{\pi}} + \frac{a\, S_3}{\bar{a}\, \overset{3}{\pi}} \qquad (4)$$

Similarly, the equation of the point x_{124} is,

$$x_{124} = a - \frac{a S_3}{\bar{a}\,{}^3\Pi} + \frac{\bar{a}}{a\,{}^3\Pi} \tag{5}$$

and

$$\bar{x}_{124} = \bar{a} - \frac{\bar{a}}{a\,{}^3\Pi} + \frac{a S_3}{\bar{a}\,{}^3\Pi} \tag{6}$$

where ${}^3\Pi = (1+t_1)(1+t_2)(1+t_4)$; $S_3 = t_1 t_2 t_4$

And so, in general, we have as the equation of x_{ijk}, the p_3 of a 3-line $\lambda_i, \lambda_j, \lambda_k$;

$$x_{ijk} = a - \frac{a S_3}{\bar{a}\,{}^3\Pi} + \frac{\bar{a}}{a\,{}^3\Pi} \tag{7}$$

and

$$\bar{x}_{ijk} = \bar{a} - \frac{\bar{a}}{a\,{}^3\Pi} + \frac{a S_3}{\bar{a}\,{}^3\Pi} \tag{8}$$

where ${}^3\Pi = (1+t_i)(1+t_j)(1+t_k)$; $S_3 = t_i t_j t_k$.

Symmetrizing, so as to pick up the four p_3 points $x_{123}, x_{124}, x_{134}, x_{234}$; we obtain the equation

$$x = a - \frac{a S_4 (1+t)}{\bar{a}\, t\, {}^4\Pi} + \frac{\bar{a}(1+t)}{a\,{}^4\Pi} \tag{9}$$

and

$$\bar{x} = \bar{a} - \frac{\bar{a}(1+t)}{a\,{}^4\Pi} + \frac{a S_4 (1+t)}{\bar{a}\, t\, {}^4\Pi} \tag{10}$$

where $\pi = (1+t_1)(1+t_2)(1+t_3)(1+t_4)$; $S_4 = t_1 t_2 t_3 t_4$

Adding (9) and (10) we have

$$x + \bar{x} = a + \bar{a} \tag{11}$$

which is the equation of a vertical line.

Consequently, we have another theorem, namely: Given four lines and an extra line λ; the four associated points p_3, which arise from the four 3-lines and an arbitrary line λ, lie on a line.

Cwopdzinski states this theorem.*

2-b Given five lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ as tangents to the double parabola

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t)^2} \tag{1}$$

(cusp condition being $\bar{A}_i \alpha_i^3 = A_i$); at the points t_1, t_2, t_3, t_4 and t_5 respectively, then the equation of the line λ_i $(i = 1, 2, 3, 4, 5)$ is

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t_i)(\alpha_i - t)} \tag{2}$$

* Archiv der Mathematik und Physik. Vol 1 - 1901, p 180

With five undirected lines, we can form five 4 lines for $C^5_4 = 5$. These five 4-lines are composed of $\lambda_1 \lambda_2 \lambda_3 \lambda_4$; $\lambda_1 \lambda_3 \lambda_4 \lambda_5$; $\lambda_1 \lambda_2 \lambda_4 \lambda_5$; $\lambda_1 \lambda_2 \lambda_3 \lambda_5$; $\lambda_2 \lambda_3 \lambda_4 \lambda_5$ respectively. Now, by theorem 1-b, we found, that to every 4-line $\lambda_i \lambda_j \lambda_k \lambda_l$ there is associated a point x_{ijkl}, a p_4. Consequently, to the above named 4-lines are associated the points x_{1234}, x_{1345}, x_{1245}, x_{1235}, x_{2345}. The equation of p_4 given in section 1-b, was for four lines taken as tangents to a parabola with the arbitrary line λ_1 here it will be necessary to find the equation of p_4, the four lines being taken as tangents to a double parabola

From 1-c we have the equation of the circumcenter, for three lines $\lambda_1, \lambda_2, \lambda_3$ taken as tangents to a double parabola, namely

$$x_{123} = \sum^{2} \frac{A\alpha}{\pi^3} \tag{3}$$

and its conjugate is

$$\bar{x}_{123} = -\sum^{2} \frac{A S_3}{\alpha \pi^3} \tag{4}$$

Reflecting this point in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \tag{5}$$

we obtain the point

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{A S_3}{\alpha \, \pi^3} \tag{6}$$

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A \alpha}{\pi^3} \tag{7}$$

A line $\perp$ to the tangent λ_4, and on the reflexion of the point x_{123} in the arbitrary line λ, is

$$t_4 x + \bar{x} = t_4 a + \frac{a}{\bar{a}} \sum^{2} \frac{A S_4}{\alpha \, \pi^3} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A \alpha}{\pi^3} \tag{8}$$

Now interpolating, so as to pick up three four perpendiculars, we obtain the equation

$$t x + \bar{x} = t a + \frac{a}{\bar{a}} \sum^{2} \frac{A S_4 (\alpha - t)}{\alpha \, \pi^4} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A \alpha (\alpha - t)}{\pi^4} \tag{9}$$

which is a point, namely p_4,

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{A S_4}{\alpha \, \pi^4} + \frac{\bar{a}}{a} \sum^{2} \frac{A \alpha}{\pi^4} \tag{10}$$

$\pi^4 = (\alpha_i - t_1)(\alpha_i - t_2)(\alpha_i - t_3)(\alpha_i - t_4)$; $S_4 = t_1 t_2 t_3 t_4$

Interpolating so as to pick up these five p_4 points, we have

$$x = a - \frac{a}{\bar{a}} \sum_1^2{}' \frac{A S_5 (\alpha - t)}{\alpha t \overset{5}{\Pi}} + \frac{\bar{a}}{a} \sum_1^2{}' \frac{A \alpha (\alpha - t)}{\overset{5}{\Pi}} \tag{11}$$

This is the equation of a conic, and its conjugate equation is

$$\bar{x} = \bar{a} - \frac{\bar{x}}{a} \sum_1^2{}' \frac{A \alpha^2 (\alpha - t)}{\overset{5}{\Pi}} + \frac{a}{\bar{a}} \sum_1^2{}' \frac{A S_5 (\alpha - t)}{t \overset{5}{\Pi}} \tag{12}$$

where $\overset{5}{\Pi} = (\alpha_i - t_1)(\alpha_i - t_2)(\alpha_i - t_3)(\alpha_i - t_4)(\alpha_i - t_5)$; $S_5 = t_1 t_2 t_3 t_4 t_5$

Hence we have the theorem:

Given five lines and an est. a line λ, the five associated points p_4, which arise from the five 4-lines and a conic tang. line λ, lie on an ellipse.

2-C Given six lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ as tangents to the double parabola

$$x = \sum^2 \frac{A_i}{(\alpha_i - t)^2} \tag{1}$$

(cusp condition being $\bar{A}_i \alpha_i^3 = A$); at the points t_1, t_2, t_3

t_4, t_5, t_6 respectively, then the equation of the tangent λ_i $(i = 1, 2, 3 \ldots 6)$ is

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t_i)(\alpha_i - t)} \tag{2}$$

With six undirected lines, we can form six 5-lines, for $C^6_5 = 6$. From theorem 1-c, we found that to every 5-line and an arbitrary line, there is associated a point p_5. Therefore to six lines and an arbitrary line, there will be associated six points p_5. The equation of the point x_{12345} is

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{A S_5}{\alpha \prod^{5}} + \frac{\bar{a}}{a} \sum^{2} \frac{A \alpha^2}{\prod^{5}} \tag{3}$$

Symmetrizing, so as to pick up the six points, namely, $x_{12345}, x_{12346}, x_{12456}, x_{13456}, x_{12356}, x_{23456}$, we have

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{A S_6 (\alpha - t)}{\alpha t \prod^{6}} + \frac{\bar{a}}{a} \sum^{2} \frac{A \alpha^2 (\alpha - t)}{\prod^{6}} \tag{4}$$

which is the map-equation of an ellipse. Its conjugate equation is

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A \alpha^3 (\alpha - t)}{\prod^{6}} - \frac{a}{\bar{a}} \sum^{2} \frac{A S_6 (\alpha - t)}{t \prod^{6}} \tag{5}$$

Hence the theorem:

Given 6 lines and an extra line λ; the six associated points p_5, which arise from the six 5 lines and an arbitrary line λ, lie on an ellipse

2-y. In general, we might say:

Given n lines and an extra line λ; the n associated points p_{n-1}, which arise from the n "(n-1)-lines" and an arbitrary line λ, lie on an ellipse.

Directed Lines

§5. First Chain of Theorems

I-a Suppose we have four directed lines L_1, L_2, L_3, L_4, given as tangents to a paracatroid at the pts, t_1, t_2, t_3, and t_4, then since the equation of the paracatroid is

$$t^4 - t^3 x + \mu t^2 - t\bar{x} + 1 = 0 \qquad (1)$$

(where μ is real); the equations of L_1, L_2, L_3, L_4 will be respectively

$$t_1^4 - t_1^3 x + \mu t_1^2 - t_1 \bar{x} + 1 = 0 \qquad (2)$$

$$t_2^4 - t_2^3 x + \mu t_2^2 - t_2 \bar{x} + 1 = 0 \tag{3}$$

$$t_3^4 - t_3^3 x + \mu t_3^2 - t_3 \bar{x} + 1 = 0 \tag{4}$$

$$t_4^4 - t_4^3 x + \mu t_4^2 - t_4 \bar{x} + 1 = 0 \tag{5}$$

The* incenter of the lines L_1, L_2, L_3 is

$$x_{123} = t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3} \tag{6}$$

and its conjugate is

$$\bar{x}_{123} = 1/t_1 + 1/t_2 + 1/t_3 + t_1 t_2 t_3 \tag{7}$$

The reflexion of the point x_{123} in the line λ, namely

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \tag{8}$$

is

$$x = a - \frac{a}{\bar{a}}\left[1/t_1 + 1/t_2 + 1/t_3 + t_1 t_2 t_3\right] \tag{9}$$

and

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}\left[t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3}\right] \tag{10}$$

We now desire to drop a perpendicular from the reflexion of x_{123} to the line L_4; since a perpendicular to L_4 is of the form

$$t_4^2 x - \bar{x} = t_4^2 c - \bar{c} \tag{11}$$

*Hodgson - Joseph E : Orthocentric Properties of the Plane Directed n-Line : Abstract of a Dissertation - J. H. U. 1910 - page 203

in order that, this perpendicular pass through the reflexion of x_{123} in the line λ, (11) becomes,

$$t_4^2 x - \bar{x} = t_4^2 \left[x - \frac{a}{\bar{a}} \left(\frac{S_2}{S_3} + S_1 \right) \right] - \left[\bar{x} - \frac{\bar{a}}{a} \left(S_1 + \frac{1}{S_3} \right) \right] \qquad (12)$$

In order, to interpolate, so as to be able to pick up the four perpendiculars, namely, the one from the reflexion of x_{234} to L_1; from the reflexion of x_{134} to L_2, etc, we will introduce the following symmetric functions:

$$S_1 = t_1 + t_2 + t_3 \qquad \sigma_1 = t_1 + t_2 + t_3 + t_4$$

$$S_2 = t_1 t_2 + t_1 t_3 + t_2 t_3 \qquad \sigma_2 = t_1 t_2 + t_1 t_3 + t_1 t_4 + t_2 t_3 + t_2 t_4 + t_3 t_4$$

$$S_3 = t_1 t_2 t_3 \qquad \sigma_3 = t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4$$

$$\sigma_4 = t_1 t_2 t_3 t_4$$

$$\therefore \; S_1 = \sigma_1 - t_4$$

$$S_2 = \sigma_2 - t_4 \sigma_1 - t_4^2$$

$$S_3 = \sigma_4 | t_4$$

Now then, making use of the above symmetric functions (12) can be written as

$$t_4^2 x - \bar{x} = t_4^2 a - \frac{a}{\bar{a}} \left[\frac{(t_4^2 \sigma_2 - t_4^3 \sigma_1 + t_4^4) t_4}{\sigma_4} + \frac{t_4 \sigma_4^2}{\sigma_4} \right] - \left[\bar{a} - \frac{\bar{a}}{a} \left(S_1 + \frac{1}{S_3} \right) \right] \qquad (13)$$

But, $$t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4 = 0 \tag{14}$$

$$\therefore\ t^4 - \sigma_1 t^3 + \sigma_2 t^2 = \sigma_3 t - \sigma_4 \tag{15}$$

Making use of this relation in (13); we obtain the equation

$$t_4^2 x - \bar{x} = t_4^2 a - \left[\frac{a t_4^2 \sigma_3 - a t_4 \sigma_4 + a t_4 \sigma_4^2}{\bar{a}\ \sigma_4}\right] - \left[\bar{a} - \frac{\bar{a}}{a}\left(S_1 + {}'S_3\right)\right] \tag{16}$$

Thus (16) is the equation of a line which passes thru the reflexion of x_{123}, and is perpendicular to L_4, which for convenience, we will call R_4. There will be four such lines, namely R_1, R_2, R_3, R_4 which will be obtained from the original four directed lines L_1, L_2, L_3, L_4.

Symmetrizing so as to pick up these four lines, we have

$$t^2 x - \bar{x} = t^2 a - \left[\frac{a t^2 \sigma_3 - a t \sigma_4 + a t \sigma_4^2}{\bar{a}\ \sigma_4}\right] - \left[\bar{a} - \frac{\bar{a}}{a}\left(\sigma_1 - t + \frac{t}{\sigma_4}\right)\right] \tag{17}$$

which is a circle.

As a result, we have the theorem:

Given four lines and an extra line λ; if we reflect the incenters of the four 3-lines in an arbitrary

line λ, and then drop perpendiculars from these reflections to the remaining line; these four perpendiculars will touch a circle.

I-b. We will next consider the five directed lines L_1, L_2, L_3, L_4, L_5 as tangents to a paracstroid, at the points t_1, t_2, t_3, t_4, t_5 respectively. L_i (where $i = 1,2,3,4,5$) will be expressed by the equation

$$t_i^4 - t_i^3 x + \mu t_i^2 - t_i \bar{x} + 1 = 0 \qquad (1)$$

where μ is real.

As given in section I-a, the incenter of the 3-line L_1, L_2, L_3 is

$$x_{123} = t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3} \qquad (2)$$

If now instead of taking three directed lines, we take four directed lines, namely, L_1, L_2, L_3, L_4, then we will be able to form four 3-lines; and since there is an incenter associated with each 3-line, there will be four incenters associated with the

four directed lines. Interpolating, so as to find up these four incenters, we obtain the equation

$$x = \sigma_1 - t + \frac{t}{\sigma_4} \tag{3}$$

where $\sigma_1 = t_1 + t_2 + t_3 + t_4$ and $\sigma_4 = t_1 t_2 t_3 t_4$. Now (3) is the equation of a circle whose center is

$$x_{1234} = \sigma_1 \tag{4}$$

We will now proceed to reflect this center in an arbitrary line, say

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \tag{5}$$

Doing this, we obtain as the reflexion of x_{1234}

$$x = a - \frac{a}{\bar{a}}\left[\frac{\sigma_3}{\sigma_4}\right] \tag{6}$$

and its conjugate equation

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}\left[\sigma_1\right] \tag{7}$$

As in the previous case, we desire to drop a perpendicular from the reflexion of x_{1234} to the remaining line L_5. The equation of such a perpen-

dicular is

$$t_5^2 x - \bar{x} = t_5^2 \left[a - \frac{\bar{x}}{\bar{a}} \left(\frac{\sigma_3}{\sigma_4} \right) \right] - \left[\bar{a} - \frac{\bar{x}}{a} \left(\sigma_1 \right) \right] \qquad (8)$$

But in this case, since we have five directed lines, we will obtain five perpendiculars similar to the one expressed by equation (8). In order to interpolate, we will have to introduce the following symmetric functions:

$$\sigma_1 = t_1 + t_2 + t_3 + t_4 \qquad P_1 = \sum^{5} t_1$$

$$\sigma_2 = \sum^{6} t_1 t_2 \qquad P_2 = \sum^{10} t_1 t_2$$

$$\sigma_3 = \sum^{4} t_1 t_2 t_3 \qquad P_3 = \sum^{10} t_1 t_2 t_3$$

$$\sigma_4 = t_1 t_2 t_3 t_4 \qquad P_4 = \sum^{5} t_1 t_2 t_3 t_4$$

$$P_5 = t_1 t_2 t_3 t_4 t_5$$

$$\therefore \sigma_1 = P_1 - t_5$$

$$\sigma_2 = P_2 - P_1 t_5 + t_5^2$$

$$\sigma_3 = P_3 - t_5 \sigma_2$$

$$\sigma_4 = P_5 \big|_{t_5}$$

But since the "P's" can be considered as the sym-

metric functions of a quintic in t, we obtain the relation

$$t^5 - P_1 t^4 + P_2 t^3 - P_3 t^2 + P_4 t - P_5 = 0 \tag{9}$$

From which, we have

$$t_5^2 \Gamma_3 = t_5^2 P_3 - t_5^3 P_2 = t_5^2 P_3 - t_5^3 P_2 + t_5^4 P_1 - t_5^5 \tag{10}$$

and also, that

$$t_5^2 P_3 - t_5^3 P_2 + t_5^4 P_1 - t_5^5 = t_5 P_4 - P_5 \tag{11}$$

Making use of the relations that exist among these two sets of symmetric functions, (8) can be written as

$$t_5^2 x - \bar{x} = t_5^2 u - \frac{a}{\bar{u}}\left[\frac{(t_5^2 P_3 - t_5^3 \Gamma_2)\, t_5}{P_5}\right] - \left[\bar{u} - \frac{\bar{u}}{\bar{a}}(\Gamma_1)\right] \tag{12}$$

or applying (11) to (12) we obtain

$$t_5^2 x - \bar{x} = t_5^2 u - \frac{u}{\bar{a}}\left[\frac{P_4 t_5^2 - P_5 t_5}{P_5}\right] - \left[\bar{u} - \frac{\bar{u}}{a}(\Gamma_1)\right] \tag{13}$$

Interpreting, so as to bring up the five perpendiculars, we have the equation

$$t^2 x - \bar{x} = t^2 u - \frac{u}{\bar{u}}\left[\frac{t^2 P_4 - t P_5}{P_5}\right] - \left[\bar{u} - \frac{\bar{u}}{u}(P_1 - t)\right] \tag{14}$$

which is a circle.

Consequently, we have the following theorem: Given five lines and an extra line, if we reflect the centric points of the five 4-lines in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, these five perpendiculars will touch a circle.

I-g In general, we might say:
Given n lines and an extra line, if we reflect the centric points of the n "$(n-1)$-lines" in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line; these n perpendiculars will touch a circle.

§6. Second Chain of Theorems

II-a Let us consider five directed lines L_1, L_2, L_3, L_4, L_5 as tangents to a parastroid, at the points t_1, t_2, t_3, t_4, t_5 respectively; then L_i (where $i = 1, 2, 3, 4, 5$) can be expressed as

$$t_i^4 - t_i^3 x + t_i^2 \mu - t_i \bar{x} + 1 = 0 \qquad (1)$$

where μ is real

By the previous theorem I - a, we found, that to every four directed lines and an set a line λ, there is associated a circle; and since there are five directed lines, we will have five 4-lines, and therefore, five circles associated with the five directed lines. The equation of one of these circles is

$$t^2 x - \bar{x} = t^2 a - \left[\frac{a t^2 \sigma_3 - a t \sigma_4 + a t \sigma_4^2}{\bar{a}\ \sigma_4}\right] - \left[\bar{a} - \frac{\bar{a}}{a}\left[\sigma_1 - t + \frac{t}{\sigma_4}\right]\right] \qquad (2)$$

Symmetrizing so as to pick up the five circles, we obtain

$$t^2 x - \bar{x} = t^2 a - \left[\frac{a t^2}{\bar{a}}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} + \frac{1}{t_5} - \frac{1}{T}\right)\right] + \frac{a t}{\bar{a}} - \frac{a t\ \sigma_5}{\bar{a}\ T}$$

$$- \bar{a} + \frac{\bar{a}}{a}\left[\sigma_1 - t - T + \frac{t\,T}{\sigma_5}\right] \qquad (3)$$

where $\sigma_1 = t_1 + t_2 + t_3 + t_4 + t_5$

$\sigma_5 = t_1 t_2 t_3 t_4 t_5$

For convenience, let us denote σ_5 by T, then (3) be-

comes

$$t^2 x - \bar{x} = t^2 a - \left[\frac{at^2}{\bar{a}}\left(\frac{\Gamma_4}{T} - \frac{1}{T}\right)\right] + \frac{t}{\bar{a}} - \frac{at}{\bar{a}}\frac{\Gamma}{T} - \bar{a} +$$

$$\frac{\bar{a}}{a}\left[\Gamma_1 - t - T + t\frac{\Gamma}{T}\right] \qquad (4)$$

or

$$t^2 x - \bar{x} = t^2 a - \left[\frac{t^2}{\bar{a}}\left(\frac{\Gamma_4}{T} - \frac{1}{T}\right)\right] + \cancel{\frac{at}{a}} - \cancel{\frac{at}{a}} - \bar{a} + \frac{\bar{a}}{a}\left[\Gamma_1 - \cancel{t} - T + \cancel{t}\right] \qquad (5)$$

Now then let $T = t$, and (5) becomes

$$t^2 x - \bar{x} = t^2 a - \left[\frac{at(\Gamma_4 - 1)}{\bar{a}}\right] - \bar{a} + \frac{\bar{a}}{a}\left[\Gamma_1 - t\right] \qquad (6)$$

or, $$a\bar{a}(t^2 x - \bar{x}) = a^2 \bar{a} t^2 - a^2 t(\Gamma_4 - 1) - \bar{a}^2 a + \bar{a}^2(\Gamma_1 - t) \qquad (7)$$

$$a\bar{a}(t^2 x - \bar{x}) = a^2 \bar{a} t^2 - a^2 t \Gamma_4 + a^2 t - \bar{a}^2 t + \bar{a}^2 \Gamma_1 - \bar{a}^2 a \qquad (8)$$

$$a\bar{a}(t^2 x - \bar{x}) = a^2 \bar{a} t^2 - t(a^2 \Gamma_4 - a^2 + \bar{a}^2) + \bar{a}^2(\Gamma_1 - a) \qquad (9)$$

$$t^2 x - \bar{x} = a t^2 - \frac{t(a^2 \Gamma_4 - a^2 + \bar{a}^2)}{a\bar{a}} + \frac{\bar{a}(\Gamma_1 - a)}{a} \qquad (10)$$

But $t = T = \Gamma_5$

$$\therefore \quad x - \frac{\bar{x}}{\Gamma_5^2} = a - \frac{(a^2 \Gamma_4 - a^2 + \bar{a}^2)}{\Gamma_5 \, a\bar{a}} + \frac{\bar{a}(\Gamma_1 - a)}{\Gamma_5^2 a} \qquad (11)$$

this is a self conjugate equation, and is the

equation of a line, the line which touches the five circles arising from the five 4-lines.

Hence we have the theorem:

Given five lines and an extra line, the five circles which arise from the five 4-lines and an arbitrary line λ, all touch the same line.

II-b. Let us consider six directed lines $L_1, L_2, L_3, L_4, L_5, L_6$ as tangents to a cyclogen, at the points $t_1, t_2, t_3, t_4, t_5, t_6$ respectively, then L_i (where $i = 1, 2, 3, 4, 5, 6$) can be expressed as

$$t_i^6 - c\, t_i^5 + x\, t_i^4 - u\, t_i^3 + \bar{x}\, t_i^2 - \bar{c}\, t_i + 1 = 0 \qquad (1)$$

where $i = 1, 2, 3, 4, 5, 6$ and u is real.

By a previous theorem I-b, we found out that to every five directed lines and an arbitrary line λ, there is associated a circle, and since there are six 5-lines, which arise from six directed lines; there will be six circles associated with the six directed lines

The equation of one of these circles is

$$t^2 x - \bar{x} = t^2 a - \frac{a}{\bar{a}}\left[\frac{t^2 P_4 - t P_5}{P_5}\right] - \left[\bar{a} - \frac{\bar{a}}{a}(P_1 - t)\right] \qquad (2)$$

Symmetrizing so as to take up these six circles, we obtain the equation

$$t^2 x - \bar{x} = t^2 a - \frac{a t^2}{\bar{a}}\left[\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} + \frac{1}{t_5} + \frac{1}{t_6} - \frac{1}{T}\right] + \frac{a t}{\bar{a}} - \bar{a}$$

$$+ \frac{\bar{a}}{a}\left[\sigma_1 - t - T\right] \qquad (3)$$

where $\sigma_1 = t_1 + t_2 + t_3 + t_4 + t_5 + t_6$; $\sigma_6 = t_1 t_2 t_3 t_4 t_5 t_6$

For convenience, let us denote σ_6 by T, then (3) becomes

$$t^2 x - \bar{x} = t^2 a - \frac{t^2}{\bar{a}}\left[\frac{\sigma_5}{T} - \frac{1}{T}\right] + \frac{t}{\bar{a}} - \bar{a} + \frac{\bar{a}\sigma_1}{a} - \frac{\bar{a} t}{a} - \frac{\bar{a} T}{a} \qquad (4)$$

$$t^2 x - \bar{x} = t^2 a - \frac{a t^2}{\bar{a}}\left[\frac{\sigma_5}{T} - \frac{1}{T}\right] + \frac{a^2 t}{a\bar{a}} - \bar{a} + \frac{\bar{a}\sigma_1}{a} - \frac{\bar{a}^2 t}{a\bar{a}} - \frac{\bar{a} T}{a} \qquad (5)$$

Now, then, let $T = t$, and (5) becomes

$$t^2 x - \bar{x} = t^2 a - \left[\frac{a t}{\bar{a}}\left(\sigma_5 - 1\right)\right] + \frac{a^2 t}{a\bar{a}} - \frac{\bar{a}^2 t}{a\bar{a}} - \bar{a} + \frac{\bar{a}}{a}(\sigma_1 - t) \qquad (6)$$

$$a\bar{a}(t^2 x - \bar{x}) = t^2 a^2 \bar{a} - a^2 t(\sigma_5 - 1) + a^2 t - \bar{a}^2 t - \bar{a}^2 a + \bar{a}^2(\sigma_1 - t) \qquad (7)$$

$$a\bar{a}(t^2x-\bar{x}) = a^2\bar{a}t^2 + t(2a^2-2\bar{a}^2-a^2\sigma_5) + \bar{a}^2(\sigma_1-a) \quad (8)$$

$$t^2x-\bar{x} = at^2 + \frac{t(2a^2-2\bar{a}^2-a^2\sigma_5)}{a\bar{a}} + \frac{\bar{a}(\sigma_1-a)}{a} \quad (9)$$

If $t = T = \sigma_6$

$$x - \frac{\bar{x}}{\sigma_6^2} = a + \frac{(2a^2-2\bar{a}^2-a^2\sigma_5)}{\sigma_6\, a\bar{a}} + \frac{\bar{a}(\sigma_1-a)}{\sigma_6^2\, a} \quad (10)$$

this is a self-conjugate equation, and is ∴ the equation of a line, the line which touches the ex circles arising from the six 5-lines

Consequently we have the theorem:

Given six lines and an extra line; the ex circles which arise from the six 5-lines and an arbitrary line λ, all touch the same line.

And in general, we can say:

II-y- Given n lines and an extra line; the "circles which arise from the n $(n-1)$-lines" and an arbitrary line λ, all touch the same line

Theorems I-a; I-b etc; II-a; II-b; etc. all apply to discrete lines

Vita

Flora Dobler Dutton was born in Baltimore, Maryland, June 1890. She was educated in the public schools of Baltimore, and after graduating from the Western High School, entered Goucher College and graduated in 1912. In 1915, she entered the Johns Hopkins University, as a graduate student and pursued courses in mathematics, education and statistics. In January 1920, she was enrolled as a special student, in the School of Hygiene and Public Health - Johns Hopkins University.

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